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Arnol'd's second stability theorem for the equivalent barotropic model

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Arnol'd's second stability theorem is proved for arbitrary perturbations of the potential vorticity field δq and the circulation(s) $\delta\gamma$. The formal stability condition is essentially the same as that for $\delta\gamma \equiv 0$, which is much easier to obtain. Similarly, the condition obtained assuming $\overline{\delta q} \equiv 0$ (the overbar denoting a horizontal average) is found to be also valid for $\overline{\delta q} \not\equiv 0$. It is argued that a Lyapunov functional that is extreme only on the sheet of constant Casimirs (and other integrals of motion) also proves stability for perturbations off the sheet, even though its second variation may not be sign definite for general perturbations. This conjecture is illustrated by means of a very simple mechanical problem: a point particle subject to the action of a central force. For the case of Phillips' problem in a periodic channel, formal stability conditions on the isovortical sheet coincide with the criteria obtained from normal modes analysis.

1. Proof of the theorem

In a recent paper (Ripa 1992, hereinafter denoted by *R92*) I worked out the Hamiltonian structure of the quasi-geostrophic multi-layer model, and derived Arnol'd's first and second sufficient stability theorems for that system. The proof of Arnol'd's second theorem is incorrect (for reasons pointed out below) even though the end result is right: let me give a correct derivation here, as well as an improvement of the stability criteria presented in *R92*.

Consider the $1\frac{1}{2}$-layer model (a more complicated vertical structure is no major problem, as will be shown below) in a domain D with boundaries ∂D_j. The state space variables are the potential vorticity q and the circulations γ_j, whose evolution is determined by

$$\partial_t q = -\partial_x \psi \partial_y q + \partial_y \psi \partial_x q, \quad \dot{\gamma}_j = 0, \tag{1 a, b}$$

where the streamfunction ψ is uniquely determined by

$$\left.\begin{aligned} (\nabla^2 - \mu^2)\psi &= q - f \quad && @\, D, \\ \oint_j \nabla\psi\cdot\boldsymbol{n} &= \gamma_j && @\, \partial D_j, \\ \boldsymbol{n}\times\nabla\psi &= 0 && @\, \partial D, \end{aligned}\right\} \tag{2}$$

μ is the inverse of the deformation radius, and $\oint_j(.)$ denotes the path integral of (.) along ∂D_j. The limit $\mu \to 0$ is rather special, as is discussed below.

Let me define the following notation

$$\left.\begin{aligned}\langle\ldots\rangle &:= \iint_D \mathrm{d}^2x(\ldots),\\ [\psi^a,\psi^b] &:= \langle\nabla\psi^a\cdot\nabla\psi^b+\mu^2\psi^a\psi^b\rangle,\end{aligned}\right\} \tag{3}$$

for ψ^a and ψ^b arbitrary functions in D, subject to $\boldsymbol{n}\times\nabla\psi = 0 \,@\, \partial D$ (e.g. the total energy is given by $\frac{1}{2}[\psi, \psi]$). From (2) and (3) it follows that

$$[\psi^a,\psi^b] = \sum_j \psi_j^a\gamma_j^b - \langle\psi^a(q-f)^b\rangle, \tag{4}$$

or similarly swapping a and b. $\sum_j$ denotes a sum over the disconnected parts of the boundary, ∂D_j, and the subscript j means the value of the variable on ∂D_j. (In all cases where that symbol is used, this value is independent of position along the contour, e.g. $\psi = \psi_j$ on account of the last equation in (2).)

Writing the dynamical fields as $q = Q+\delta q$ and $\gamma = \Gamma+\delta\gamma$, where (Q,Γ) represents some basic state, formal stability of the latter is assured by finding a conserved functional of state, whose lowest-order variation is sign definite. Making $\mathscr{S} = \mathscr{H}+\mathscr{C}$, where $\mathscr{H}$ is the Hamiltonian and $\mathscr{C}$ an appropriately chosen Casimir such that $\delta\mathscr{S} = 0$, a sufficient condition for $\delta^2\mathscr{S} > 0$ ($\delta^2\mathscr{S} < 0$) constitutes Arnol'd's first (second) formal stability theorem (McIntyre & Shepherd 1987; *R92*). Both Γ and $\delta\gamma$ are time independent, in virtue of (1b), but a value of $\delta\gamma \neq 0$ represents a contribution to the velocity field in the interior which affects the time evolution of $\delta q(x,t)$. Consequently, a non-vanishing $\delta\gamma$ represents in principle a non-trivial complication of the stability problem.

Consider the conserved Lyapunov functional

$$\mathscr{S}[q,\gamma_j] = \tfrac{1}{2}[\psi,\psi]+\langle F(q)\rangle+\sum_j c_j\gamma_j, \tag{5}$$

with $F(q)$ and c_j arbitrary. Its first variation from the basic state is

$$\delta\mathscr{S} = \langle(F'(Q)-\Psi)\,\delta q\rangle+\sum_j(c_j-\Psi_j)\,\delta\gamma_j;$$

in order for this to vanish for all perturbations $(\delta q,\delta\gamma)$, it is necessary that $\Psi = F'(Q)$ and $\Psi_j\,(=\Psi \,@\, \partial D_j) = c_j$ (this requires the basic state to be steady). The total variation of $\mathscr{S}$ is then

$$\Delta\mathscr{S} = \langle\sigma(Q,\delta q)\rangle+\tfrac{1}{2}[\delta\psi,\delta\psi], \tag{6a}$$

where $$\sigma(Q,\delta q) := (\Delta-\delta)\,F(q) = \tfrac{1}{2}\Psi'(Q)\,\delta q^2+\tfrac{1}{6}\Psi''(Q)\,\delta q^3+\ldots. \tag{6b}$$

If $\Psi'(Q)$ is positive (Arnol'd's first theorem) then formal stability is easily proved; nonlinear stability requires $\sigma/\delta q^2$ to be bounded between two positive numbers.

The difficulty with the second theorem, $\Psi' < 0$, lies in that $[\delta\psi,\delta\psi]$ is a functional of both δq and $\delta\gamma$. Let me attack this problem with a decomposition of $\delta\psi$ in the form

$$\delta\psi(x,t) = \delta\psi^q(x,t)+\delta\psi^\gamma(x), \tag{7a}$$

to be specified below, which makes

$$\Delta\mathscr{S} = \langle\sigma(Q,\delta q)\rangle+\tfrac{1}{2}[\delta\psi^q,\delta\psi^q]+[\delta\psi^q,\delta\psi^\gamma]+\tfrac{1}{2}[\delta\psi^\gamma,\delta\psi^\gamma]. \tag{7b}$$

In *R92* I chose (7a) such that $[\delta\psi^q,\delta\psi^\gamma] \equiv 0$; this requires $\delta\psi^\gamma$ to be a linear function of $\delta\gamma_j-\delta\hat{\gamma}_j$, where $\delta\hat{\gamma}_j$ is 'produced' by δq (see (3.8b) and (3.10) in that paper). However, the statement in *R92* (p. 387) in the sense that $[\delta\psi^\gamma,\delta\psi^\gamma]$ is time independent is incorrect, since even though $\delta\gamma_j$ is constant, $\delta\hat{\gamma}_j$ can be time dependent. To correct that error, consider instead the following definition of decomposition (7*a*)

$$\begin{cases} (\nabla^2-\mu^2)\,\delta\psi^q = \delta q, \quad \oint_j \nabla\delta\psi^q\cdot\boldsymbol{n} = 0, \quad \boldsymbol{n}\times\nabla\delta\psi^q = 0 \,@\, \partial D, & (8a) \\ (\nabla^2-\mu^2)\,\delta\psi^\gamma = 0, \quad \oint_j \nabla\delta\psi^\gamma\cdot\boldsymbol{n} = \delta\gamma_j, \quad \boldsymbol{n}\times\nabla\delta\psi^\gamma = 0 \,@\, \partial D. & (8b) \end{cases}$$

It is very easy to prove uniqueness of this decomposition; existence will be shown for a particular example, the periodic channel, by means of constructing a suitable expansion basis. From (4) and (8) one finds

$$[\delta\psi^q,\delta\psi^q] = -\langle\delta\psi^q\,\delta q\rangle,\ [\delta\psi^\gamma,\delta\psi^\gamma] = \textstyle\sum_j \delta\psi_j^\gamma\,\delta\gamma_j,$$

and

$$[\delta\psi^q,\delta\psi^\gamma] = -\langle\delta\psi^\gamma\,\delta q\rangle = \sum_j \delta\psi_j^q\,\delta\psi_j; \tag{8c}$$

notice that $[\delta\psi^q+\delta\psi^\gamma,\delta\psi^q+\delta\psi^\gamma] = -\langle\delta\psi\,\delta q\rangle+\sum_j\delta\psi_j\,\delta\gamma_j$, as required by (4).

For this decomposition, $[\delta\psi^\gamma,\delta\psi^\gamma]$ is time independent, so it does not represent a problem for a stability theorem. Expression (7*b*) can be rearranged in the form

$$\sum_j(\rho_j\,\delta\gamma_j-\delta\psi_j^q/\rho_j)^2-(\delta\psi_j^q/\rho_j)^2-[\delta\psi^q,\delta\psi^q]-2\langle\sigma(Q,\delta q)\rangle = M, \tag{9}$$

for any $\rho_j \neq 0$, where $M := 2\Delta\mathscr{S}-[\delta\psi^\gamma,\delta\psi^\gamma]-\sum_j(\rho_j\,\delta\gamma_j)^2$ is time independent by construction. In order to prove nonlinear stability I assume that it is possible to make the bounds

$$[\delta\psi^q,\delta\psi^q] \leqslant (\lambda^2+\mu^2)^{-1}\langle\delta q^2\rangle, \tag{10a}$$

$$(\delta\psi_j^q)^2 \leqslant \nu_j^2\langle\delta q^2\rangle, \tag{10b}$$

$$(\lambda^2+\mu^2)^{-1} < a \leqslant -2\sigma(Q,\delta q)/\delta q^2 \leqslant A, \tag{10c}$$

for some λ^2, ν_j^2 and a pair of positive numbers a and A. Using (9) and (10) it is possible to prove that

$$M \leqslant A\langle\delta q^2\rangle+\sum_j(\rho_j\,\delta\gamma_j-\delta\psi_j^q/\rho_j)^2 \tag{11a}$$

and

$$W\langle\delta q^2\rangle+\sum_j(\rho_j\,\delta\gamma_j-\delta\psi_j^q/\rho_j)^2 \leqslant M, \tag{11b}$$

where $W := a-(\lambda^2+\mu^2)^{-1}-\sum_j\nu_j^2/\rho_j^2$ is chosen positive (with ρ_j^2 large enough). Using (11*a*) at $t=0$, it follows that the left-hand side of (11*b*) (whose square root qualifies as a norm $\|\delta q,\delta\psi\|$) is bounded by A/W times its initial value, i.e. the basic flow is nonlinearly stable.

Of the three bounds in (10), the last one is a condition on the structure of the basic state (it is, in fact, stability condition (3.13) from *R92*), whereas the first two represent a property of the domain D and equation (8*a*). Notice that the particular value of ν_j^2 in (10*b*) does not appear in the stability condition (10*c*). This is not the case of parameter λ^2, from (10*a*), which does play an important role in the stability condition. If one imposes $\delta\gamma_j \equiv 0$ to start with then (10*b*) is not needed at all. A sufficient condition for formal stability is obtained simply by replacing the bounds in (10*c*) by

$$\mathrm{d}\Psi/\mathrm{d}Q < -(\lambda^2+\mu^2)^{-1}; \tag{12}$$

this is condition (3.12) in *R92*.

I will show next how inequalities (10*a*) and (10*b*) can be proved, by the explicit evaluation of λ^2 and ν_j^2 for the example of the periodic zonal channel. Even through a value of $\nu_j^2 < \infty$ was used here to prove normed stability for $\delta\gamma_j \neq 0$, bound (10b) is not really needed to evaluate the stability range of a family of basic flows; this statement will be clarified with examples in the periodic channel.

2. The zonal periodic channel

Consider the domain $[0, L_x] \otimes [0, L_y]$, with periodic boundary conditions in x. The decomposition in (8) can be done by expanding the perturbation in a suitable basis. An appropriate one for $\delta\psi^q$ is

$$\chi_{nm}(\boldsymbol{x}) := \begin{cases} \cos(l_m y), & n = 0, \quad m \geqslant 0, \\ \sin(l_m y) \times \begin{cases} \cos(k_n x), \\ \sin(k_n x), \end{cases} & n = \pm 1, \pm 2, \ldots, \quad m \geqslant 1, \end{cases} \tag{13a}$$

where $k_n := 2n\pi/L_x$ and $l_m := m\pi/L_y$. A basis for the expansion of $\delta\psi^\gamma$ is given by

$$\vartheta_\pm(\boldsymbol{x}) = \exp(\pm\mu y), \tag{13b}$$

or any independent combinations of these two functions. (The basis used in the inappropriate decomposition proposed in *R92* is given by (13a) replacing $\cos(l_m y)$ by $\sin(l_m y)$ in the $n = 0$ case.)

Using basis (13a) to expand δq yields

$$\delta q(\boldsymbol{x}, t) = \sum_{nm} A_{nm}(t)\, \chi_{nm}(\boldsymbol{x}) \Rightarrow \delta\psi^q(\boldsymbol{x}, t) = -\sum_{nm} \kappa_{nm}^{-2} A_{nm}(t)\, \chi_{nm}(\boldsymbol{x}), \tag{14}$$

where $\kappa_{nm}^2 := k_n^2 + l_m^2 + \mu^2$, from which it follows that

$$\langle \delta q^2 \rangle = \tfrac{1}{4} L_x L_y \sum_{nm} A_{nm}^2 (1 + \delta_{n0}(1 + 2\delta_{m0}))$$

and

$$[\delta\psi^q, \delta\psi^q] = \tfrac{1}{4} L_x L_y \sum_{nm} \kappa_{nm}^{-2} A_{nm}^2 (1 + \delta_{n0}(1 + 2\delta_{m0})).$$

Consequently, inequality (10a) is satisfied, with

$$\lambda^2 + \mu^2 \equiv \kappa_{00}^2 = \mu^2. \tag{15a}$$

Moreover, at both boundaries one finds

$$\delta\psi^q = -\sum_m \kappa_{0m}^{-2} A_{0m} \quad (y = 0), \quad \delta\psi^q = -\sum_m (-1)^m \kappa_{0m}^{-2} A_{0m} \quad (y = L_y),$$

and since $\langle \delta q^2 \rangle \geqslant L_x L_y \sum_m A_{0m}^2$, the bound (10$b$) is easily proved using Schwartz inequality, i.e.

$$(\textstyle\sum_m \kappa_{0m}^{-2} A_{0m})^2 \leqslant \sum_m \kappa_{0m}^{-4} \sum_m A_{0m}^2,$$

which yields

$$\nu^2 \propto \sum_{m=0}^{\infty} \frac{1}{[(m\pi/L_y)^2 + \mu^2]^2}. \tag{15b}$$

I have proved bounds (10a) and (10b) for the periodic channel; consequently (10c)/(15a) is a nonlinear stability condition.

Notice that (15b) is not valid for $\mu = 0$. This case (barotropic with 'rigid lid') is special in the sense that the streamfunction is defined modulo addition of a constant and the potential vorticity and circulations are not independent, i.e.

$$\mu = 0 \Rightarrow \langle q - f \rangle = \sum_j \gamma_j.$$

In order to deal with the $\mu = 0$ case, I will derive next a new stability condition, valid for any μ, which in fact is an improvement over condition $(10c)/(15a)$.

First of all, notice that $\langle \delta q \rangle$ is time independent (it is one of the Casimirs), hence the part of the streamfunction related to it can be moved from $\delta\psi^q(\boldsymbol{x}, t)$ to $\delta\psi^\gamma(\boldsymbol{x})$, i.e. (8) can be modified as

$$\begin{cases} (\nabla^2 - \mu^2)\,\delta\psi^q = \delta q - \overline{\delta q}, & \oint_j \nabla\delta\psi^q \cdot \boldsymbol{n} = 0, \quad \boldsymbol{n}\times\nabla\delta\psi^q = 0 \ @\ \partial D, & (8a)' \\ (\nabla^2 - \mu^2)\,\delta\psi^\gamma = \overline{\delta q}, & \oint_j \nabla\delta\psi^\gamma \cdot \boldsymbol{n} = \delta\gamma_j, \quad \boldsymbol{n}\times\nabla\delta\psi^\gamma = 0 \ @\ \partial D. & (8b)' \end{cases}$$

where $\overline{\delta q} := \langle \delta q \rangle / \langle 1 \rangle$ is the (constant) horizontal average of δq. Equation (4) indicates that $(8c)$ should be replaced by

$$[\delta\psi^q, \delta\psi^\gamma] = \sum_j \delta\psi^q_j\,\delta\gamma_j - \langle \delta\psi^q\,\overline{\delta q} \rangle.$$

However $(8a)'$ implies that $\langle \delta q - \overline{\delta q} \rangle = 0 = -\mu^2 \langle \delta\psi^q \rangle$ and therefore if $\mu^2 \neq 0$ it must be $\langle \delta\psi^q \rangle = 0$, whereas if $\mu^2 = 0$ one may choose $\langle \delta\psi^q \rangle = 0$ since in this case the streamfunction is defined modulo the addition of a constant. The end result is that $(8c)$ does hold true. The analysis proceeds more or less as before† and the same stability condition $(10c)$ is obtained, the key difference being that $\delta\psi^q$ in $(10a)$ and $(10b)$ has now a vanishing horizontal average.

For the example of the periodic channel, subtracting $\overline{\delta q}$ from $\delta q(\boldsymbol{x}, t)$ to calculate $\delta\psi^q$ (i.e. making $\langle \delta\psi^q \rangle = 0$) corresponds to simply eliminating χ_{00} from the basis $(13a)$ and adding $\vartheta_0 := 1$ to the basis $(13b)$. Consequently, the bounds $(10a)$ and $(10b)$ now correspond to

$$\lambda^2 + \mu^2 \equiv \kappa_{01}^2 = (\pi/L_y)^2 + \mu^2, \tag{15a)'}$$

$$\nu^2 \propto \sum_{m=1}^{\infty} \frac{1}{[(m\pi/L_y)^2 + \mu^2]^2}, \tag{15b)'}$$

which are better than $(15a)$ and $(15b)$ in the sense that the bounds $(10a)$ and $(10b)$ are more restrictive with the new values of λ^2 and ν^2.

Moreover, $(15b)'$ does not diverge as $\mu \to 0$. In this limit, one has only two functions for the expansion of $\delta\psi^\gamma$, say $\vartheta = y - \frac{1}{2}L_y$, $y^2 - \frac{1}{3}L_y^2$ instead of the three functions ($\exp(\pm\mu y)$ and 1) used for $\mu \neq 0$. Loss of one expansion function is consistent with the constraint $\langle \delta q \rangle = \sum_j \delta\gamma_j$ valid for $\mu = 0$.

Comparing stability conditions $(10c)/(15a)'$ and $(10c)/(15a)$ there are two important points to make: First, $(10c)/(15a)'$ is stronger than $(10c)/(15a)$, because it corresponds to a larger class of proved stable states (recall that these criteria are sufficient for stability or necessary for instability). Secondly, if one assumes $\overline{\delta q} = 0$, which represents a peculiar class of perturbations, albeit preserved by the dynamics, condition $(10c)/(15a)'$ is easily obtained, instead of $(10c)/(15a)$. However, the former is also valid for $\overline{\delta q} \neq 0$.

3. Extension of the theorem

In sum, I have shown how sign definiteness of $\delta^2(\mathscr{H} + \mathscr{C})$ on the sheet $(\delta\gamma, \overline{\delta q}) = 0$, i.e. condition $(12)/(15)'$, guarantees also formal stability on nearby sheets, in which $(\delta\gamma, \overline{\delta q}) \neq 0$. A simple geometrical interpretation is the following: the dynamics

† An expression of the form $A\,\delta q^2 + B(\delta q - \overline{\delta q})^2$ has to be written as $(A+B)(\delta q - C\,\overline{\delta q})^2 + D$.

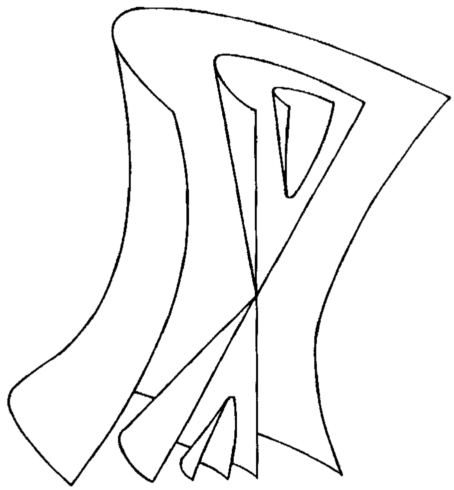

FIGURE 1. Topology of the surfaces of constant pseudoenergy in state space. Conservation of the other integrals of motion requires the system to move in some 'horizontal' plane: even though the central point is not an absolute extremum, it is a stable point, because the intersections with horizontal planes all have an elliptic point.

requires the motion to lie on some sheet $(\delta\gamma, \overline{\delta q}) = \text{const}$, and – excluding pathological cases – if $\delta^2(\mathscr{H}+\mathscr{C})$ has an elliptical point on $(\delta\gamma, \overline{\delta q}) = 0$, it will be so on nearby sheets, i.e. $(\delta\gamma, \overline{\delta q})$ sufficiently small (see figure 1). Therefore the motion is formally stable even though $\delta^2(\mathscr{H}+\mathscr{C})$ may not be sign definite for all perturbations.

Of course, if pressed to define 'pathological' I will end up in a circular argument. Let me instead clarify this idea (which is not a proof) using a very simple example: the motion of a particle on the plane, subject to a central potential $\phi(r)$; the azimuthal angle, which is a cyclic variable, is factored out from state space, thereby playing the role of the 'inner' symmetries of the Eulerian fluid problems. The system can be described by the Hamiltonian

$$H(r,u,l) = \tfrac{1}{2}u^2 + \tfrac{1}{2}l^2 r^{-2} + \phi(r) \tag{16a}$$

and Poisson brackets $\quad \{r,u\} = 1, \quad \{r,l\} = \{u,l\} = 0, \qquad (16b)$

such that $\dot{\eta} = \{\eta, H\}\,\forall\,\eta(r,u,l)$. The angular momentum l is here a Casimir,

$$\{\eta, l\} = 0\,\forall\,\eta(r,u,l),$$

and one of the coordinates. However, the latter property is not important in what follows: the power of Hamiltonian formalism lies in its manifest covariance under change of variables (more on this below).

Defining

$$\left.\begin{aligned} r(t) &= R(t)+\delta r(t), \\ u(t) &= U(t)+\delta u(t), \\ l &= L+\delta l, \end{aligned}\right\} \tag{17}$$

and requiring $\delta(H-\Omega l) = 0$, for some constant Ω, yields the steady solution

$$R = \text{const}, \quad U = 0, \quad L = R^2\Omega, \quad \phi'(R) = R\Omega^2. \tag{18}$$

In order to study the stability of this solution, one calculates the second differential of $H-\Omega l$, namely

$$\delta^2(H-\Omega l) = \delta u^2 + \omega^2\,\delta r^2 + R^{-2}\,\delta l^2 - 4\Omega R^{-1}\,\delta r\,\delta l \tag{19a}$$

$$\delta^2(H-\Omega l) \equiv \delta u^2 + (\omega^2 - 4\Omega^2)\,\delta r^2 + (R^{-1}\,\delta l - 2\Omega\,\delta r)^2 \tag{19b}$$

$$\delta^2(H-\Omega l) \equiv \delta u^2 + \omega^2(\delta r - 2R^{-1}\omega^{-2}\,\delta l)^2 - \omega^{-2}(\omega^2-4\Omega^2)\,R^{-2}\,\delta l^2, \tag{19c}$$

where I have used $L = R^2\Omega$ and defined

$$\omega^2 := \phi''(R) + 3\Omega^2 \equiv \phi_e''(R), \tag{20}$$

with $\phi_e(r) := \phi(r) + L^2/2r^2$. From (19*a*) it follows that if

$$\omega^2 > 4\Omega^2 \tag{21}$$

[i.e. $\phi''(R) > \phi'(R)/R$] then $\delta^2(H-\Omega l)$ is positive definite for arbitrary perturbations, and therefore the solution (18) is stable.

However, by solving explicitly the linearized equations for $(\delta r, \delta u, \delta l)$ it is found that $\omega^2 > 0$ [i.e. $\phi''(R) > -3\phi'(R)/R$] is sufficient for stability; (19) shows this to be the condition for $\delta^2(H-\Omega l)$ to be positive definite not for arbitrary perturbations, but for those on the sheet $\delta l = 0$. Notice that (19*c*) implies that if $\omega^2 > 0$ and $\delta l \neq 0$, then $\delta^2(H-\Omega l)$ has an elliptical point at $(u = 0, r = R + 2R^{-1}\omega^{-2}\delta l)$; that is why the solution (18) is stable. The figure shows the surfaces of $H-\Omega l$ in state space (the u and r axes are horizontal and the l axis is vertical) for a basic state such that $0 < \omega^2 < 4\Omega^2$. The cones correspond to $\delta^2(H-\Omega l) \equiv 0$, whereas in their interior (exterior) this integral of motion is negative (positive), as indicated by (19*c*).

Of course, in this case it is possible to construct an integral of motion which has an absolute maximum at (18), even for $0 < \omega^2 < 4\Omega^2$; this is given by $H + F(l)$, where $F(l)$ is arbitrary, save for $F'(L) = -\Omega$ and $F''(L) > \omega^{-2}(\omega^2 - 4\Omega^2)R^{-2}$. However, the point here is that it is enough to prove sign definiteness of $\delta^2(H-\Omega l)$ on $\delta l = 0$ to guarantee stability even for perturbations with $\delta l \neq 0$. This is an analogy for the results of the last section, with l playing the role of γ and/or $\bar{q}$, and $H-\Omega l$ being the analogue of the Lyapunov functional (5). One could also replace the latter by

$$\mathcal{S}[q, \gamma_j] = \tfrac{1}{2}[\psi, \psi] + \langle F(q)\rangle + \sum_j F_j(\gamma_j),$$

with $F_j'(\Gamma_j) = c_j$ and $F_j''(\Gamma_j)$ negative and large enough to make $\delta^2\mathcal{S}$ negative definite. However, as long as formal stability is concerned, it is enough to prove sign definiteness on a sheet of constant integrals of motion.

It might be argued that this analogy is not a good one, because l is conserved and it is one of the coordinates. However, the same exercise can be easily done with the Hamiltonian

$$H(r, u, v) := \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2 + \phi(r) \tag{22a}$$

and the Poisson brackets

$$\{r, u\} = 1, \quad \{r, v\} = 0, \quad \{u, v\} = v/r. \tag{22b}$$

The second differential (19) can be written in terms of $(\delta r, \delta u, \delta v)$ but the reasoning would be less clear. In this representation $l \equiv vr$ is a Casimir but not one of the coordinates, i.e. l is the analogue to the Casimirs $\langle F(q)\rangle + \sum_j c_j \gamma_j$ of the last section which, alas, are not state variables.

4. Discussion

Proving sign definiteness of one integral of motion on the sheet where all other integrals of motion are kept fixed is enough to guarantee formal stability to small perturbations of arbitrary shape, i.e. off the sheet. Consider again the $1\frac{1}{2}$-layer quasi-geostrophic model on a periodic zonal channel. In order to prove the stability of a steady and parallel basic flow,

$$Q(y) = f + \Psi''(y) - \mu^2\Phi(y), \quad U(y) = -\Psi'(y), \tag{23}$$

the most general Lyapunov functional is given by an arbitrary combination of pseudoenergy and pseudomomentum, i.e. $\mathscr{H}-\alpha\mathscr{M}+\mathscr{E}$ for any α, where the Casimir $\mathscr{C}$ is chosen so that $\delta(\mathscr{H}-\alpha\mathscr{M}+\mathscr{C})=0$. Formal stability is guaranteed if $\delta^2(\mathscr{H}-\alpha\mathscr{M}+\mathscr{C})$ is proved to be sign definite for perturbations such that $\langle F'(Q)\,\delta q\rangle = 0\,\forall\, F(q)$. This constraint on the perturbation implies that the x-average of δq must vanish $\forall\, y$. That means that it is enough to prove sign definiteness of $\delta^2(\mathscr{H}-\alpha\mathscr{M}+\mathscr{C})$ for δq given by the expansion (14) with $n \neq 0$, i.e. instead of (15a) or (15a)$'$ one has

$$\lambda^2+\mu^2 \equiv \kappa_{11}^2 = (2\pi/L_x)^2+(\pi/L_y)^2+\mu^2, \qquad (15a)''$$

which makes of (10c) or (12) a stronger stability condition, because the value of λ^2 is larger than those of (15a)$'$ and (15a).

The extended Arnol'd's second theorem then reads

$$\frac{\beta-U''(y)+\mu^2U(y)}{U(y)-\alpha} < (2\pi/L_x)^2+(\pi/L_y)^2+\mu^2 \qquad (24)$$

for all y and some α, which represents an improvement over equation (4.1b) of *R92*, on account of the term $(2\pi/L_x)^2$. Take for instance

$$U(y) = U_0\sin(\Lambda y)-\beta/\mu^2, \qquad (25)$$

the necessary condition for instability derived using $\alpha = -\beta/\mu^2$ in (24) is

$$\Lambda^2 > (2\pi/L_x)^2+(\pi/L_y)^2, \qquad (26)$$

which is more restrictive (for a finite length channel) than equation (4.3) of *R92*. Criterion (26) coincides with the normal mode one; see (4.5) in *R92*.

Finally, let me point out that a more complicated vertical structure does not represent a major problem for the stability theorems developed here. If a common bound is sought for $\mathrm{d}\Psi_j/\mathrm{d}Q_j$ in all layers, then a weak form of the stability condition is obtained, that is, either (10c) or (12) with μ^{-1} equal to the largest deformation radius in a vertical normal modes spectrum. For instance, we can rederive the stability condition for Phillips' problem (treated in *R92*) which now reads

$$f_0^2(H_1+H_2)/(g'H_1H_2) < (2\pi/L_x)^2+(\pi/L_y)^2, \qquad (27)$$

and is less demanding than condition (5.1) of that paper. Moreover, following the procedure in §5 of *R92*, the stronger condition

$$2f_0^2/g'(H_1H_2)^{\frac{1}{2}} < (2\pi/L_x)^2+(\pi/L_y)^2 \qquad (28)$$

is now obtained, which guarantees that $\Delta\mathscr{H}$ is negative definite on the $\langle F(q)\rangle$ = constant and γ_j = constant sheet. This formal stability condition is stronger than condition (5.5) of *R92* on account of the term $(2\pi/L_x)^2$. Moreover, (28) coincides with the normal modes stability condition; see (5.9)† and following in *R92*.

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Note added in proof: Mu Mu & T. G. Shepherd have submitted a paper to *Geophys. Astrophys. Fluid Dyn.* with a similar theorem. Their derivation corresponds to subtracting the contribution of $\overline{\delta q}$ to $\delta\Psi^q$, i.e. to (15a)$'$ in the case of the periodic channel.

† That equation should read $\Xi^2 = \kappa_1\kappa_2(\kappa_1\kappa_2-1)$.

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